ANALYTIC CONTINUATION

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DECLARATION

I declare that the topic "**ANALYTIC CONTINUATION** " for completion for my master degree has not been submitted in any other institution or university for the award of any other degree or diploma.

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THESIS CERTIFICATE

This is to certify that the project report entitled **Analytic Continuation** submitted by **Kishore Chandra Dalai** to the National Institute of Technology Rourkela, Orissa for the partial fulfilment of requirements for the degree of master of science in Mathematics is a bonafide record of review work carried out by her under my supervision and guidance. The contents of this project, in full or in parts, have not been submitted to any other institute or university for the award of any degree or diploma.

(Bappaditya Bhowmik)

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Kishore Chandra Dalai

ABSTRACT

In this thesis, we study the following topics in complex analysis:- (1) Montel's theorem. (2) Riemann Mapping theorem. (3) Analytic continuation. We also study the celebrated Schwarz Reflection Principle and the Monodromy theorem.

Contents

NOTATION

English Symbols

$\mathbb{C}$	the complex plane.
$\mathbb{D}$	the unit disk $\{z \in \mathbb{C} : \|z\| < 1\}$.
$\overline{B}(a, R)$	the closed ball center at a and radius R.
$\mathbb{H}(G)$	set of analytic functions in G.
$A \subset B$	A is a proper subset of B.
$\mathcal{S}$	class of normalized analytic univalent functions

CHAPTER 1

INTRODUCTION

Complex variable is a branch of mathematics which has something for all mathematician. In addition, to having application to other parts of analysis, it can exactly postulation to be an antecedent of many areas of mathematics. In fact, in this thesis, our purpose is to focuss on some topics in complex analysis such as spaces of analytic functions, normal family, spaces of meromorphic functions and celebrated Riemann mapping theorem. We finally review some theory of analytic continuation including the Schwarz reflection principle and the Monodromy theorem.

CHAPTER 2

SPACES OF ANALYTIC FUNCTIONS

In this chapter, we put a metric on the family of all analytic functions on a fixed domain G, and discuss compactness and convergence in this metric space. We also focus on spaces of analytic functions and give proof of the celebrated *Hurwitz's theorem*, *normal families*, *Montel's theorem*.

DEFINITION 2.1 (Spaces of analytic functions). Let G is an open set in $\mathbb{C}$ and (Ω, d) is a complete metric space then designate by $C(G, \Omega)$ the set of all continuous functions from G to Ω .

PROPOSITION 2.2. *Let G is open in $\mathbb{C}$ then there is a sequence $\{K_n\}$ of compact subsets of G such that $G = \bigcup_{n=1}^{\infty} K_n]$. Moreover, the sets $\{K_n\}$ can be chosen to satisfy the following conditions:*

(a) $K_n \subset int(K_{n+1})$.

(b) $K \subset G$ *and* K *compact implies* $K \subset K_n$ *for some* n.

(c) *Every component of* $\mathbb{C}_\infty - K_n$ *contains a component of* $\mathbb{C}_\infty - \mathbb{G}$.

PROOF. For each positive integer n, let $K_n = \{z : |z| < n\} \cap \{z : d(z, \mathbb{C} - G) \geq \frac{1}{n}\}$. Since K_n is bounded and it is intersection of two closed subsets of $\mathbb{C}$. So K_n is compact. Now consider the set $M = \{z : |z| < n+1\} \cap \{z : d(z, \mathbb{C} - \mathbb{G}) \geq \frac{1}{n+1}\}$ is open. Hence $K_n \subset M$ and $M \subset K_{n+1}$. So $K_n \subset \text{int }(K_n)$. G is an open set, so $G = \bigcup_{n=1}^{\infty} K_n$. Then we can get $G = \bigcup_{n=1}^{\infty} \text{int}(K_n)$. If K is compact subset of G, then the set $\text{int}(K_n)$ form an open cover of K. So $K \subset K_n$ for some n. Now we wish to prove that every component of $\mathbb{C}_\infty - K_n$ contains a component of $\mathbb{C}_\infty - G$. The unbounded component of $\mathbb{C}_\infty - K_n$

must contain ∞. So the component of $\mathbb{C}_\infty - G$ which contains ∞. Also the unbounded component contains $\{z : |z| > n\}$. So if $\mathbb{D}$ is a bounded component ,it contains a point z with $d(z, \mathbb{C} - G) < \dfrac{1}{n}$. According to definition this gives a point w in $\mathbb{C} - G$ with $|z - w| < \dfrac{1}{n}$. But then $z \in B\left(w, \dfrac{1}{n}\right) \subset \mathbb{C}_\infty - K_n$; since disks are connected and z is in the component $\mathbb{D}$ of $\mathbb{C}_\infty - k_n$, $B\left(w, \dfrac{1}{n}\right) \subset \mathbb{D}$. If $\mathbb{D}_1$ is the component of $\mathbb{C}_\infty - \mathbb{D}$ that contains w it follows that $\mathbb{D}_1 \subset \mathbb{D}$. $\square$

Proposition 2.3. *If* $G = \bigcup_{n=1}^{\infty} K_n$ *where each* K_n *is compact and* $K_n \subset int(K_{n+1})$,*define* $\rho_n(f,g) = sup\{d(f(z), g(z)) : z \in K\}$ *for all functions* f *and* g *in* $C(G, \Omega)$. *Also define,* $\rho(f,g) = \Sigma_{n=1}^{\infty} (\frac{1}{2})^n \frac{\rho_n(f,g)}{1+\rho_n(f,g)}$. *Prove that* $(C(G,\Omega), \rho)$ *is a metric space.*

Proof. (i)$\rho(f,g) \geq 0$,$\forall f, g \in C(G,\Omega)$.Since $\rho_n(f,g) = \sup\{d(f(z), g(z)) : z \in K_n\} \geq 0$.(ii)$G = \bigcup_{n=1}^{\infty} K_n$ gives that $f = g$,then $\rho(f,g) = 0$.(iii) $\rho(f,g) = \rho(g,f)$, $\forall f, g \in C(G,\Omega)$. Since

$$\begin{aligned}
\rho_n(f,g) &= sup\{d(f(z), g(z)) : z \in K_n\} \\
&= sup\{d(g(z), f(z)) : z \in K_n\} \\
&= \rho_n(g,f)
\end{aligned}$$

(iv) To prove triangle inequality we first establish the following inequality. Let $0 \leq \alpha \leq beta$, then $\alpha + \alpha\beta \leq \beta + \alpha\beta$.Dividing both sides by $(1+\alpha)$ $(1+\beta)$,we obtain

$$\frac{\alpha}{1+\alpha} \leq \frac{\beta}{1+\beta} \tag{2.1}$$

Now, for all $f, g, h \in C(G,\Omega)$, we have ,

$$\begin{aligned}
0 \leq \; & sup\{d(f(z), g(z)) : z \in K_n\} \\
\leq \; & sup\{d(f(z), h(z)) : z \in K_n\} + sup\{d(h(z), g(z)) : z \in K_n\}
\end{aligned}$$

So from above equation(2.1) it is follows that

$$\begin{aligned} & \frac{sup\{d(f(z),g(z)) : z \in \ K_n\}}{1+ \ sup\{d(f(z),g(z)) : z \in \ K_n\}} \\ \leq & \frac{sup\{d(f(z),h(z)) : z \in \ K_n\} + \ sup\{d(h(z),g(z)) : z \in \ K_n\}}{1+ \ sup\{d(f(z),h(z)) : z \in \ K_n\} + \sup\{d(h(z),g(z)) : z \in \ K_n\}} \end{aligned}$$

Multiplying both sides by 2^{-n} and summing w.r.t n, we get,

$$\Sigma_{n=1}^{\infty}(\frac{1}{2})^n \frac{\rho_n(f,g)}{1+\rho_n(f,g)} \leq \Sigma_{n=1}^{\infty}(\frac{1}{2})^n \frac{\rho_n(f,h)}{1+\rho_n(f,h)} + \Sigma_{n=1}^{\infty}(\frac{1}{2})^n \frac{\rho_n(h,g)}{1+\rho_n(h,g)}$$

i.e.

$$\rho(f,g) \leq \rho(f,h) + \rho(h,g)$$

Hence, $(C(G,\Omega),\rho)$ is a metric space. □

LEMMA 2.4. *Let the metric ρ be defined as in $\rho(f,g) = \Sigma_{n=1}^{\infty}(\frac{1}{2})^n \frac{\rho_n(f,g)}{1+\rho_n(f,g)}$. If $\epsilon > 0$ is given then there is a $\delta > 0$ and a compact set $K \subset G$ such that for f and g in $C(G,\Omega)$; $\sup\{d(f(z),g(z)) : z \in K\} < \delta \Rightarrow \rho(f,g) < \epsilon$. Conversely, if $\delta > 0$ and a compact set K are given, there is an $\epsilon > 0$ such that for f and g in $C(G,\Omega)$, $\rho(f,g) < \epsilon \Rightarrow \sup\{d(f(z),g(z)) : z \in \ K\} < \delta$.*

PROOF. Now we will prove $\sup\{d(f(z),g(z)) : z \in K\} < \delta \Rightarrow \rho(f,g) < \epsilon$. Let $\epsilon > 0$ is fixed and p be a positive number such that $\sum_{n=p+1}^{\infty}\left(\frac{1}{2}\right)^n < \frac{1}{2}\epsilon$. Put $K = K_p$. Choose $\delta > 0$ such that $0 \leq t \leq \delta$ gives $\frac{t}{1+t} < \frac{1}{2}\epsilon$. Suppose $f,g \in C(G,\Omega)$ such that $\sup\{d(f(z),g(z)) : z \in K\} < \delta$. Since $K_n \subset K_p = K$ for $1 \leq n \leq p$, $0 < \rho_n(f,g) < \delta$.
So

$$\frac{\rho_n(f,g)}{1+\rho_n(f,g)} < \left(\frac{1}{2}\right)\epsilon.$$

Here,

$$\begin{aligned}
\rho(f,g) &= \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n\left(\frac{\rho_n(f,g)}{1+\rho_n(f,g)}\right)\\
&= \left(\sum_{n=1}^{p}\left(\frac{1}{2}\right)^n\left(\frac{\rho_n(f,g)}{1+\rho_n(f,g)}\right)\right)+\left(\sum_{n=p+1}^{\infty}\left(\frac{1}{2}\right)^n\left(\frac{\rho_n(f,g)}{1+\rho_n(f,g)}\right)\right)\\
&= \sum_{n=1}^{p}\left(\frac{1}{2}\right)^n\left(\frac{\epsilon}{2}\right)+\frac{\epsilon}{2}<\frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.
\end{aligned}$$

Now, we wish to prove that $\rho(f,g)<\epsilon \Rightarrow \sup\{d(f(z),g(z)) : z\in K\}<\delta$. Let K and δ are given, Since $G=\bigcup\limits_{n=1}^{\infty} k_n=\bigcup\limits_{n=1}^{\infty} intK_n$ and K is compact there is an integer $p\geq 1$ such that $K\subset K_p$; this gives $\rho_p(f,g)\geq \sup\{d(f(z),g(z)) : z\in K\}$. Let $\epsilon>0$ be chosen so that $0\leq s\leq 2^p\epsilon$.

$$\Rightarrow \frac{s}{1-s}<\frac{2^p\epsilon}{1-2^p\epsilon}=\delta \Rightarrow \frac{s}{1-s}<\delta$$
$$\Rightarrow 0\leq t\leq \delta \Rightarrow \frac{t}{1+t}<\frac{s}{1+s}=2^p\epsilon$$

So if $\rho_p(f,g)<\epsilon \Rightarrow \dfrac{\rho_p(f,g)}{1+\rho_p(f,g)}<\epsilon$
$\Rightarrow \rho_p(f,g)<\delta \Rightarrow \sup\{d(f(z),g(z)) : z\in K\}<\delta$. This completes the proof. □

PROPOSITION 2.5. *(a) A set $\mathbb{O}\subset (C(G,\Omega),\rho)$ is open iff for each f in $\mathbb{O}$ there is a compact set K and a $\delta>0$ such that $\mathbb{O}\supset\{g : d(f(z),g(z))<\delta, z\in K\}$.*
(b) A sequence $\{f_n\}$ in $(C(G,\Omega),\rho)$ converges to f iff $\{f_n\}$ converges to f uniformly on all compact subsets of G.

PROOF. (a)Let $\mathbb{O}$ is open and $f\in\mathbb{O}$, then for some $\epsilon>0$, $\{g : \rho(f,g)<\epsilon\}\subset\mathbb{O}$. According to lemma (3.3) there exist $\delta>0$ and a compact set K such that

$$\{g : d(f(z),g(z))<\delta; z\in K\}\subset\mathbb{O}.$$

Conversely, if for each $f\in\mathbb{O}$ there is a compact set K and $\delta>0$ such that

$$\{g : d(f(z),g(z))<\delta; z\in K\}\subset\mathbb{O},$$

then from the second part of the previous lemma (3.3); we get $\mathbb{O}$ is open.
(b) Given that f_n in $(C(G,\Omega))$ converges to f. Now we have to prove that f_n converges to

f uniformly on all compact subsets of G, i.e to prove f_n converges to f $\forall f$ and $\forall$ $z \in G$. Let for given $\epsilon > 0$,

$$\begin{aligned}
& \rho(f_n, f) < \epsilon \\
\Rightarrow \quad & \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n \left(\frac{\rho_n(f_n, f)}{1 + \rho_n(f_n, f)}\right) < \epsilon \\
\Rightarrow \quad & \rho_n(f_n, f) < \epsilon \\
\Rightarrow \quad & \sup\{d(f_n(z), f(z)), z \in K_n\} < \epsilon.
\end{aligned}$$

So f_n converges to f $\forall$ f *and* $\forall$ $z \in G$ i.e f_n converges to f uniformly on all compact subsets of G. Conversely, given that f_n converges to f uniformly, $d(f_n, f) < \epsilon \forall f$, then $\sup\{d(f_n, f) : z \in k_n\}$, $\rho_n(f_n, f) < \epsilon \Rightarrow \rho(f_n, f) < \epsilon$. So f_n converges to f. □

PROPOSITION 2.6. *$C(G, \Omega)$ is a complete metric space.*

PROOF. Let f_n be a cauchy sequence in $C(G, \Omega)$. If we restrict our domain of the sequence of functions gives a Cauchy sequence f_n to $C(K, \Omega)$ for every compact sets K in G.i.e, for every $\delta > 0$ there is an integer N such that $\sup\{d(f_n(z), f_m(z)) : z \in K\} < \delta$ for $n, m \geq N$. In particular $\{f_n\}$ is a Cauchy sequence in Ω; so there is a point $f(z)$ in Ω such that $f(z) = \lim f_n(z)$. This gives a function $f : G \to \Omega$; it must be shown that f is continuous and $\rho(f_n, f) \to 0$. Let K be compact and fixed $\delta > 0$; choose N so that $\sup\{d(f_n(z), f_m(z)) : z \in K\} < \delta$ should satisfy for $n, m > N$. If z is arbitrary in K but fixed then there is an integer $m > N$ so that $d(f(z), f_m(z)) < \delta$. But then $d(f(z), f_n(z)) < 2\delta$ for all $n \geq N$. Since N does not depend on z, this gives $\sup\{d(f(z), f_n(z)) : z \in K\} \to 0$ as $n \to \infty$. Hence, f_n converges to f uniformly on every compact set in G. In particular converges on all closed balls contained in G. Since uniform limit of a sequence of continuous function is continuous, we see that f is continuous at each point of G. Also $\rho(f_n, f) \to 0$ according to proposition $(3.4(b))$. □

DEFINITION 2.7 (Normal families). A set $\mathbb{F} \subset C(G, \Omega)$ is normal if each sequence in $\mathbb{F}$ has a subsequence which converges to a function f in $C(G, \Omega)$.

DEFINITION 2.8 (Locally bounded). A set $\mathbb{F} \subset \mathbb{H}(G)$ is locally bounded if for each point a in G there are constants M and $r > 0$ such that for all f in $\mathbb{F}$, $|f(z)| \leq M$, for $|z - a| < r$ i.e $\sup\{f(z) : |z - a| < r, f \in \mathbb{F}\} < \infty$.

DEFINITION 2.9 (Equicontinuous at a point). A set $\mathbb{F} \subset C(G, \Omega)$ is equicontinuous at a point $z_0 \in G$ iff for every $\epsilon > 0$ such that for $|z - z_0| < \delta$, $d(f(z), f(z_0)) < \epsilon$ for every f in $\mathbb{F}$.

DEFINITION 2.10 (Equicontinuous over a set). $\mathbb{F}$ is equicontinuous over a set $E \in G$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that for z and z_0 in $\mathbb{F}$ and $|z - z_0| < \delta$, $d(f(z), f(z_0)) < \epsilon$ f $\forall f \in \mathbb{F}$.

THEOREM 2.11 (Arzela-Ascoli theorem). *A set $\mathbb{F} \subset C(G, \Omega)$ is normal iff the following two conditions are satisfied:*
(a) For each $z \in G$,$\{f(z) : f \in \mathbb{F}\}$ has compact closure in Ω.
(b) $\mathbb{F}$ is equicontinuous at each point of $\mathbb{G}$.

DEFINITION 2.12 (Meromorphic unction). If G is open and f is a function defined and analytic in G except for poles, then f is a meromorphic function on G.

THEOREM 2.13 (Rouche's Theorem). *Suppose f and g are meromorphic in a neighborhood of $\overline{B}(a; R)$ with no zeros or poles on the circle $\gamma = \{z : |z - a| = R\}$. If Z_f, Z_g (p_f,p_g) are the number of zeros(poles) of f, g inside γ counted according to their multiplicities and if $|f(z) + g(z)| < |f(z)| + |g(z)|$ on γ then, $Z_f - P_f = Z_g - P_g$.*

PROOF. From the hypothesis

$$|\frac{f(z)}{g(z)} + 1| < |\frac{f(z)}{g(z)}| + 1$$

on γ. If $\lambda = \dfrac{f(z)}{g(z)}$ and if λ is a positive real number, then this inequality becomes $\lambda + 1 < \lambda + 1$. This is a contradiction, hence the meromorphic function $\dfrac{f}{g}$ maps γ onto

$\Omega = \mathbb{C} - [0, \infty)$. If l is a branch of the logarithm on Ω, then $l\left(\dfrac{f(z)}{g(z)}\right)$ is well-defined primitive for $(f/g)^{'}(f/g)^{-1}$ in a neighborhood of γ.Thus

$$\begin{aligned}
&= \frac{1}{2\pi i}\int_{\gamma}(f/g)^{'}(f/g)^{-1} \\
&= \frac{1}{2\pi i}\int_{\gamma}\left[\frac{f^{'}}{f} - \frac{g^{'}}{g}\right] \\
&= (Z_f - P_f) - (Z_g - P_g).
\end{aligned}$$

So we have $Z_f - P_f = Z_g - P_g$. □

THEOREM 2.14 (Hurwitz's Theorem). *Let G be a region and suppose the sequence $\{f_n\}$ in $\mathbb{H}(G)$ converges to f. If f is not identical to zero, $\overline{B}(a;R)$ and $f(z) \neq 0$ for $|z-a| = R$, then there is an integer N such that for $n \geq N$, f and $\{f_n\}$ have the same number of zeros in $B(a;R)$.*

PROOF. Let G be a region and $\{f_n\}$ in $\mathbb{H}(G)$ converges to f. Since $f(z) \neq 0\ \forall |z-a| = R$, let $\delta = \inf\{|f(z)| : |z-a| = R\} > 0$. But $\{f_n\} \to f$ uniformly on $|z| : |z-a| = R$. So there is an integer N such that if $n \geq N$ and $|z-a| = R$, then $|f(z) - f_n(z)| < \frac{1}{2}\delta < |f(z)| \leq |f(z)| + |f_n(z)|$. According to Rouches theorem f and $\{f_n\}$ have same number of zeros in $B(a;R)$. □

THEOREM 2.15 (Montel's Theorem). *A family $\mathbb{F}$ in $\mathbb{H}(G)$ is normal iff $\mathbb{F}$ is locally bounded.*

PROOF. Suppose $\mathbb{F}$ is normal but fails to be locally bounded; then there is a compact set $K \in G$ such that $\sup\{|f(z)| : z \in k, f \in \mathbb{F}\} = \infty$, i.e there is a sequence $\{f_n\}$ in $\mathbb{F}$ such that $\sup\{|f(z)| : z \in k\} \geq n$. Since $\mathbb{F}$ is normal there is a function f in $\mathbb{H}(G)$ and a sequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$. But this gives that $\sup\{|f_{n_k}(z) - f(z)| : z \in k\} \to 0$ as $K \to \infty$. If $|f(z)| \leq M$ for z in K, $n_k \leq \sup\{|f_{n_k}(z) - f(z)| : z \in k\} + M$. Since the right hand side converges to M, so this is a contradiction. So $\mathbb{F}$ is locally bounded. conversely, suppose $\mathbb{F}$ is locally bounded. Here we use Arzela-Ascoli theorem to show $\mathbb{F}$

is normal. It can be easily shown that the first condition is satisfied. Now only we have to prove the second condition of this theorem, i.e we have to prove $\mathbb{F}$ is equicontinuous at each point of G. Let fix a point $a \in G$ and $\epsilon > 0$, so according to hypothesis $\exists r > 0$ and $M > 0$ such that $\overline{B}(a;r) \subset G$ and $|f(z)| \le M \ \forall z \in \overline{B}(a;r)$ and $\forall f \in \mathbb{F}$. Let $|z-a| < \frac{1}{2}r$ and $f \in \mathbb{F}$; then using Cauchy's formula with $\gamma(t) = a + re^{it}$, $0 \le t \le 2\pi$, we get

$$\begin{aligned} |f(a) - f(z)| &\le \frac{1}{2\pi}\left|\int_\gamma \frac{f(w)(a-z)}{(w-a)(w-z)}dw\right| \\ &\le \frac{1}{2\pi}|a-z|\left|\int_\gamma \frac{f(w)}{(w-a)(w-z)}dw\right| \end{aligned} \tag{2.2}$$

At $w = a$

$$\left|\lim_{w\to a} \frac{f(w)(w-a)}{(w-a)(w-z)}\right| = \frac{M}{(1/2)r} = \frac{2M}{r} \tag{2.3}$$

(2.4)

At $w = z$

$$\left|\lim_{w\to z} \frac{f(w)(w-z)}{(w-a)(w-z)}\right| = \frac{M}{(1/2)r} = \frac{2M}{r} \tag{2.5}$$

According to Cauchy's formula and from (2.3) and (2.5), we get from (2.2)

$$\int_\gamma \frac{f(w)}{(w-a)(w-z)}dw = 2\pi\left(\frac{2M}{r} + \frac{2M}{r}\right) = \frac{8M\pi}{r} \tag{2.6}$$

Again from (2.2) and from (2.6) we get,

$$|f(a) - f(z)| = \frac{1}{2\pi}|a-z|\frac{8M\pi}{r} = |a-z|\frac{4M}{r}$$

Let $\delta = \min\{\frac{1}{2r}, \frac{r\epsilon}{4M}\}$. So $|a-z| < \delta$. So $|f(a) - f(z)| < \epsilon \ \forall f \in \mathbb{F}$. Hence the second condition satisfied. Hence it is proved. $\square$

CHAPTER 3

SPACES OF MEROMORPHIC FUNCTIONS AND THE RIEMANN MAPPING THEOREM

In this chapter, we put a metric on extended complex plane and defined a Chordal metric. We also focus on spaces of meromorphic function functions and give proof of the celebrated *Riemann Mapping Theorem* .

1. Spaces of meromorphic functions

Let G be a region and f is a meromorphic function on G, Let $M(G)$ is the set of all continuous functions on G then consider $M(G)$ as a subset of $C(G, \mathbb{C}_\infty)$. In this section we are going to put a metric d on $\mathbb{C}_\infty$ as follows. Let $z_1, z_2 \in \mathbb{C}$,then

$$d(z_1, z_2) = \frac{2|z_1 - z_2|}{[(1 + |z_1|^2)(1 + |z_2|^2)]^{\frac{1}{2}}};$$

and for each z in $\mathbb{C}$ we define, $d(z, \infty) = \dfrac{2}{(1 + |z|^2)^{\frac{1}{2}}}$.

THEOREM 3.1. *Let f_n be a sequence in $M(G)$ and $f_n \to f$ in $C(G, \mathbb{C}_\infty)$.Then either f is meromorphic or $f \equiv \infty$.If each f_n is analytic then either f is analytic or $f \equiv \infty$.*

PROOF. Suppose there is a point a in G with $f(a) \neq \infty$ and put $M = |f(a)|$.We know from one proposition, If a is in $\mathbb{C}$ and $r > 0$ then there is a number $\rho > 0$ such that $B_\infty(a; \rho) \subset B(a; r)$. According to this proposition we can find a number $\rho > 0$ such that $B_\infty(f(a); \rho) \subset B(f(a); M)$. But since $f_n \to f$ there is an integer n_0 such that $d(f_n(a), f(a)) < \frac{1}{2}\rho$ for all $n \geq n_0$. Also $\{f, f_1, f_2, ...\}$ is compact in $\mathbb{C}(G, \mathbb{C}_\infty)$ so that it is equicontinuous.That is, there is an $r > 0$ such that $|z-a| < r$ implies $d(f_n(z), f_n(a)) < \frac{1}{2}\rho$. That gives that $d(f_n(z), f(a)) \leq \rho$ for $|z - a| \leq r$ and for $n \geq n_0$. But by choice of ρ,

$|f_n(z)| \leq |f_n(z) - f(a)| + |f(a)| \leq 2M$ for all z in $\overline{B}(a;r)$ and $n \geq n_0$. But from the formula for the metric d , we have

$$\frac{2}{1+4M^2}|f_n(z) - f(z)| \leq d(f_n(z), f(z)$$

for z in $\overline{B}(a;r)$ and $n \geq n_0$. Since $d(f_n(z), f(z)) \to 0$ uniformly for z in $\overline{B}(a;r)$. Since the sequence f_n is bounded on $B(a;r)$,f_n has no poles and must be analytic near $z = a$ for $n \geq n_0$. It follows that f is analytic in a disk about a.

Now suppose that there is a point a in $\mathbb{G}$ with $f(a) = \infty$.For function g in$\mathbb{C}(\mathbb{G}, \mathbb{C}_\infty)$ define $\frac{1}{g}$ by $(\frac{1}{g})(z) = \frac{1}{g(z)}$ if $g(z) \neq 0$ or ∞; $(\frac{1}{g})(z) = 0$ if $g(z) = \infty$; and $(\frac{1}{g})(z) = \infty$ if $g(z) = 0$.It follows that $\frac{1}{g} \in \mathbb{C}(\mathbb{G}, \mathbb{C}_\infty)$.Also since $f_n \to f$ in $\mathbb{C}(\mathbb{G}, \mathbb{C}_\infty)$.From chordal metric it is follows that $\frac{1}{f_n} \to \frac{1}{f}$ in $\mathbb{C}(\mathbb{G}, \mathbb{C}_\infty)$.Now each function $\frac{1}{f_n}$ is meromorphic on$\mathbb{G}$;so it gives a number $r > 0$ and a integer n_0 such that $\frac{1}{f_n}$ and $\frac{1}{f}$ are analytic on $B(a;r)$ for $n \geq n_0$ and $\frac{1}{f_n} \to \frac{1}{f}$ uniformly on $B(a;r)$.From Hurwitz's theorem either $\frac{1}{f} \equiv 0$ or $\frac{1}{f}$ has isolated zeros in $B(a;r)$. □

Corollary 3.2. *$M(G) \cup \{\infty\}$ is a complete metric space.*

Corollary 3.3. *$\mathbb{H}(G) \cup \{\infty\}$ is closed in $C(G, \mathbb{C}_\infty)$.*

2. The Riemann Mapping Theorem

The theorem was stated by *Bernhand Riemann* in 1851 in his Ph.D. thesis. According to Riemann Mapping theorem,any two proper simply connected domains in the plane are homeomorphic.Even though class of continuous functions are vastly larger than the class of conformal maps, it is not easy to construct a one-to-one function onto the disc,knowing only that the domains are simply connected.

Definition 3.4 (Conformally equivalent). A region G_1 is conformally equivalent to G_2 if there is analytic function $f : G_1 \to \mathbb{C}$ such that f is one-one and $f(G_1) = G_2$. $\mathbb{C}$ is not equivalent to any bounded region. By Liouville's theorem, if *f* is entire and bounded for all values of z in $\mathbb{C}$, then $f(z)$ is constant through out the plane. So constant function can not be one-one .Hence $\mathbb{C}$ is not equivalent to any bounded region.

Theorem 3.5 (Riemann Mapping Theorem). *: Given any simply connected region Ω which is not the whole plane and the point $a \in \Omega$, then there exits a unique analytic function $f(z) \in \Omega$,normalized by the conditions.*

$$(a)\quad f(a) = 0 \textit{ and } f^{'}(a) > 0 \tag{3.1}$$
$$(b)\quad f \textit{ is one} - \textit{one}.$$
$$(c)\quad f(\Omega) = \{z : |z| < 1\}$$

Proof. Suppose $\Omega \neq \mathbb{C}$ is a simply connected region.Let $z_0 \in \Omega$,We have to show that (1)Uniqueness of f having the properties in (3.1).

Let g be another analytic function which is defined by $g : \Omega \rightarrow \mathbb{C}$, which satisfies all the conditions of (3.1), then $g(a) = 0$ and $g^{'}(a) > 0$ implies that $a = g^{-1}(0)$ $(f \circ g^{-1})(0) = f(a) = 0$. Suppose f and g are two functions on Ω,then S=$f \circ g^{-1}$: $\mathbb{D} \longrightarrow \mathbb{D}$ is analytic, one-one and onto. $f \circ g^{-1}(0)$=$f(a)$=0. According to Schwarz's lemma, $(f \circ g^{-1})$ is a one-one map then there is constant c with $|c| = 1$ and $(f \circ g^{-1})(z) = cz \forall z$ then $f(z) = cg(z)$ since $f^{'}(a) > 0$, so $cg^{'}(a) > 0$ and also we have $g^{'}(a) > 0$. So c must be 1.So finally we have $f = g$, i.e, f is unique. (2) Existence:To motivate the proof of the existance of f,(2) Existence: Consider the family of functions $\mathbb{F}$ of all analytic functions f having properties (a) and (b) from (3.1) and satisfying $|f(z)| < 1$ for z in Ω. Now only we have to choose a member of $\mathbb{F}$ having property (c) of the equation (3.1). Suppose $\{K_n\}$ is a sequence of compact subsets of Ω such that $\Omega = \bigcup\limits_{n=1}^{\infty} K_n$ and $a \in K_n$ for each n, then $\{f(K_n)\}$ is a sequence of compact subsets of $\mathbb{D}$ where $\mathbb{D} = \{z : |z| < 1\}$.As n becomes larger, $\{f(K_n)\}$ becomes larger and larger and tries to fill out the disc $\mathbb{D}$. i.e, $\mathbb{D} = \bigcup\limits_{n=1}^{\infty} f(K_n)$. In a simply connected region every non vanishing analytic function has an analytic square root.So to prove existence of f, we have to prove the following lemma. In a simply connected region every non vanishing analytic function has an analytic square root. So to prove existence of f, we have to prove the following lemma.

LEMMA 3.6. *Let Ω be a region which is not the whole plane and such that every non vanishing analytic function on Ω has an analytic square root. If $a \in \Omega$, then there is an analytic function f on Ω such that*

$$(a) \quad f(a) = 0 \ and \ f^{'}(a) > 0$$

$$(b) \quad f \ is \ one - one.$$

$$(c) \quad f(\Omega) = \mathbb{D} = \{z : |z| < 1\}$$

PROOF. Define $\mathbb{F} = \{f \in \mathbb{H}(\Omega) : f \ is \ one - one, a \in \Omega f(a) = 0, f^{'}(a) > 0, f(\Omega) \subset \mathbb{D}\}$. Since $f(\Omega) \subset \mathbb{D}$, $\sup\{|f(z)| : z \in \Omega\} \leq 1$ for $f \in \mathbb{F}$. According to Montel's theorem $\mathbb{F}$ is normal if it is non-empty, i.e, we have to prove (i) $\mathbb{F} \neq \phi$ and (ii) $\mathbb{F}^{-} = \mathbb{F} \cup \{0\}$. First assume that equation (i) and (ii) hold. Consider the function $f \to f^{'}(a)$ of $\mathbb{H}(\Omega) \to \mathbb{C}$. This is a continuous function. Since $\mathbb{F}^{-}$ is compact, then there exist $f \in \mathbb{F}^{-}$ with $f^{'}(a) \geq g^{'}(a) \forall g \in \mathbb{F}$. Because $\mathbb{F} \neq \phi$ then $f \in \mathbb{F}$. Now only we have to show that $f(\Omega) = \mathbb{D}$ i.e, we have to show that (a)$f(\Omega) \subset \mathbb{D}$ and (b)$\mathbb{D} \subset f(\Omega)$. To prove equation $f(\Omega) = \mathbb{D}$; let$w$ does not belongs to $f(\Omega)$. Then the function $\dfrac{f(z) - w}{1 - \bar{w} f(z)}$ is analytic in Ω and never vanishes. By hypothesis there is an analytic function $h : \Omega \to \mathbb{C}$ such that $[h(z)]^2 = \dfrac{f(z) - w}{1 - \bar{w} f(z)}$. Since the Möbius transformation $T(\xi) = \dfrac{\xi - w}{1 - \bar{w}\xi}$ maps $\mathbb{D}$ onto $\mathbb{D}, h(\Omega) \subset \mathbb{D}$. Define $g : \Omega \to \mathbb{C}$ by

$$g(z) = \left(\frac{|h^{'}(a)|}{h^{'}(a)}\right)\left(\frac{h(z) - h(a)}{1 - \overline{h(a)} h(z)}\right).$$

Here, it is easy to see that $g(a) = 0$, and

$$g^{'}(z) = \left(\frac{|h^{'}(a)|}{h^{'}(a)}\right)\left(\frac{h^{'}(z)(1 - h(a)\overline{h(a)})}{[1 - \overline{h(a)} h(z)]^2}\right).$$

Hence

$$g^{'}(a) = \left(\frac{|h^{'}(a)|}{h^{'}(a)}\right)\left(\frac{h^{'}(a)(1 - h(a)\overline{h(a)})}{[1 - \overline{h(a)} h(a)]^2}\right) = \frac{|h^{'}(a)|}{1 - |h(a)|^2} > 0.$$

Now, $|h(a)|^2 = |-w| = |w|$ and

$$h(z)^2 = \frac{f(z-w)}{1-\bar{w}f(z)}.$$

On differentiation of the function h yields,

$$\begin{aligned}
2h(z)h^{'}(z) &= \frac{(1-\bar{w}f(z))f^{'}(z)-(f(z)-w)(-\bar{w}f^{'}(z))}{(1-\bar{w}f(z))^2} \\
&= \frac{f^{'}(z)-\bar{w}f(z)f^{'}(z)+f(z)f^{'}(z)\bar{w}-w\bar{w}f^{'}(z)}{(1-\bar{w}f(z))^2} \\
&= \frac{f^{'}(z)(1-|w|^2)}{(1-\bar{w}f(z))^2}.
\end{aligned}$$

Hence,

$$2h(a)h^{'}(a) = \frac{f^{'}(a)(1-|w|^2)}{(1-\bar{w}f(a))^2} = f^{'}(a)(1-|w|^2)\ (\because\ f(a)=0).$$

and as a result,

$$h^{'}(a) = \frac{f^{'}(a)(1-|w|^2)}{2h(a)}.$$

Now

$$\begin{aligned}
g^{'}(a) &= \frac{|h^{'}(a)|}{1-|h(a)|^2} > 0 \\
\Rightarrow g^{'}(a) &= \frac{f^{'}(a)(1-|w|^2)}{2h(a)(1-|w|)} = \frac{f^{'}(a)(1-|w|^2)}{2\sqrt{|w|}(1-|w|)} = \frac{f^{'}(a)(1+|w|)}{2\sqrt{|w|}} > f^{'}(a).
\end{aligned}$$

This gives that g is in $\mathbb{F}$ and contradicts the choice of f. So $w \in f(\Omega)$. So, $\mathbb{D} \subset f(\Omega)$. Again $f(\Omega) \subset \mathbb{D}$. So we get $f(\Omega) = \mathbb{D}$. Now we are going to prove the equations (i) and (ii). Since $\Omega \neq \mathbb{C}$, let $b \in \mathbb{C} - \Omega$ and let g be an analytic function on Ω such that $[g(z)]^2 = z - b$. If z_1 and z_2 are points in Ω, and $g(z_1) = \pm g(z_2)$, then it follows that $z_1 = z_2$. In particular g is one to one . According to Open mapping theorem there is a $r > 0$ such that $B(a, R) \subset g(\Omega)$. So there is a point z in Ω such that $g(z) \in B(-g(a); r)$ then $r > |g(z) + g(a)| = |-g(z) - g(a)|$. Since $B(a, R) \subset g(\Omega)$, so there is a w in Ω with $g(w) = -g(z)$; but $B(a, R) \subset g(\Omega)$ shows that $w = z$ which gives $g(z) = 0$. But then $z - b = [g(b)]^2 = 0$ implies that b is in Ω, a contradiction. Hence $g(\Omega) \cap \{\xi : |\xi + g(a)| < r\} = \phi$. Let U be the disk $\{\xi : |\xi + g(a)| < r\} = B(-g(a); r)$. There is a Möbius transformation T such that $T(\mathbb{C}_\infty - U^-) = \mathbb{D}$. Let $g_1 = T \circ g$; then $g_1(\Omega) \subset \mathbb{D}$. If

$\alpha = g(a)$, then let $g_2(z) = \phi_\alpha \circ g_1(z)$; so we will have that $g_2(\Omega) \subset \mathbb{D}$ and g_2 is analytic, but we also have that $g_2(a) = 0$. Now there is a complex number c with $|c| = 1$, such that $g_3(z) = cg_2(z)$ has positive derivative at $z = a$ and is therefore in $\mathbb{F}$. Here f is $g_3 = c(\phi_\alpha \circ T \circ g)$. So from this we conclude that the set $\mathbb{F}$ is nonempty. Suppose $\{f_n\}$ is a sequence in $\mathbb{F}$ and $f_n \to f$ in $H(\Omega)$. Clearly $f(a) = 0$ and since $f_n^{'}(a) \to f^{'}(a)$, then it follows that $f^{'}(a) \geq 0$. Let z_1 be an arbitrary element of Ω and put $\xi = f(z_1)$; let $\xi_n = f_n(z_1)$. Again let $z_2 \in \Omega$, $z_1 \neq z_2$ and let K be a closed disk centered at z_2 such that $z_1 \notin K$. Then $f_n(z) \to \xi_n$ never vanishes on disk K. Since f_n is one- one. But $f_n(z) - \xi n \to f(z) - \xi$ uniformly on K. According to Hurwitz's theorem gives that $f(z) - \xi$ never vanishes on K or $f(z) \equiv \xi$. If $f(z) \equiv \xi$ on K, then f is the constant function ξ throughout Ω; since $f(a) = o$ we have that $f(z) \equiv 0$. Otherwise we get that $f(z_1) \neq f(z_2)$ for $z_1 \neq z_2$; that is f is one-one. But if f is one-one then $f^{'}$ can never vanish. So $f^{'}(a) > 0$ and f is in $\mathbb{F}$. This proves the equation (ii) completely, which proves the existence of f in $\mathbb{F}$. □

Now from this above lemma we conclude that f exists with properties in equation (3.1). This completes the proof of the Riemann mapping theorem. □

CHAPTER 4

ANALYTIC CONTINUATION AND MONODROMY THEOREM

In this chapter, we define Analytic continuation and some interesting examples. We also focus on Analytic continuation along a path and give proof of celebrated *Schwarz Reflection Principle* and *Monodromy Theorem*.

1. Analytic Continuation

DEFINITION 4.1. Analytic Continuation is a technique to extend the domain of a given analytic function. Suppose f is an analytic function defined on an open subset U of the complex plane $\mathbb{C}$. If V is a larger open subset of $\mathbb{C}$ containing U and F is an analytic function defined on V such that $F(z)$=$f(z)$, then F is called an analytic continuation of f.

Example: Consider the first function f_1 is defined by the equation

$$\begin{aligned} f_1(z) &= \sum_{n=0}^{\infty} z^n \\ &= \lim_{n\to\infty} [1 + z + z^2 + \cdots + z^n] \\ &= \lim_{n\to\infty} \frac{1 - z^{n+1}}{1 - z} \\ &= \frac{1}{1-z} - \lim_{n\to\infty} \frac{z^{n+1}}{1-z} \\ &= \frac{1}{1-z} \quad \text{whenever} \quad |z| < 1. \end{aligned}$$

If $|z| > 1$, then $\lim |z|^n = \infty$ and the series diverges. Now the function $f_2(z) = \frac{1}{1-z}$ is defined and analytic in $\mathbb{C} - \{1\}$. We see that $f_2(z) = f_1(z)$, for all $|z| < 1$. Hence the function f_2 is analytic continuation of f_1.

2. Schwarz Reflection Principle

The theorem was stated by *H.A.Schwarz*. This article is about the reflection principle in complex analysis. In mathematics, Schwarz reflection principle is a way to extend the domain of definition of an analytic function of a complex variable F, which is defined on the upper half-plane and has well-defined and real number boundary values on the real axis.In that case,the putative extension of F to the rest of the complex plane is $\overline{F(\overline{z})}$. If G is a region and $G^* = \{z : \overline{z} \in G\}$ and if f is an analytic function on$G$ then$f^* : G^* \longrightarrow \mathbb{C}$ defined by $f^*(z) = \overline{f(\overline{z})}$ is also analytic. Now suppose that $G = G^*$; that is, G is symmetric with respect to the real axis. Then $g(z) = f(z) - \overline{f(\overline{z})}$ is analytic on G. Since G is connected it must be that G contains an open interval of the real line. Suppose $f(x)$ is real for all x in $G \cup \Re$. But $G \cup \Re$ has a limit point in G so that $f(z) = \overline{f(\overline{z})}$, $\forall z \in G$. The fact that f must satisfy this equation is used to extend a function defined on $G \cup \{z : Imz \geq 0\}$ to all of G.
If G is symmetric region (i.e., G=G^*) then let $G_+ = \{z \in G : Imz > 0\}$, $G_- = \{z \in G : Imz < 0\}$, and $G_0 = \{z \in G : Imz = 0\}$.

Schwarz Reflection Principle: Let G be a region such that $f : G_+ \cup G_0 \longrightarrow \mathbb{C}$ is a continuous function which is analytic on G_+ and if $f(x)$ is real for x in G_0, then there is an analytic function $g : G \longrightarrow \mathbb{C}$ such that $g(z) = f(z)$ for z in $G_+ \cup G_0$.

Proof. For z in G_- define $g(z)$=$\overline{f(\overline{z})}$ and for z in $G_+ \cup G_0$ let $g(z) = g(z)$. It is easy to see that $g : G \longrightarrow \mathbb{C}$ is continuous; it must be shown that g is analytic. It is trivial that g is analytic on $G_+ \cup G_-$, so fix a point x_0 in G_0 and let $R > 0$ with $B(x_0; R) \subset G$. It is sufficient to show that: g is analytic on $B(x_0; R)$. To do this apply Morera's Theorem.

Let T= $[a, b, c, a]$ be a triangle in $B(x_0; R)$. To show that : $\int_T f = 0$ it is sufficient to show that $\int_P f = 0$, whenever P is a triangle or a quadrilateral lying entirely in $G_+ \cup G_0$ or $G_- \cup G_0$. Assume that $T \subset G_+ \cup G_0$ and $[a, b] \subset G_0$.

Let $\triangle$ designate T together with its inside; then $g(z) = f(z)$ for all z in $\triangle$. By hypothesis f is continuous on $G_+ \cup G_0$ and so, f is uniformly continuous on $\triangle$. (since $\triangle = T \cup inside \subset G_+ \cup G_0$). So, if $\epsilon > 0$ there is a $\delta > 0$ such that when z and $z^{'} \in \triangle$ and $|z - z^{'}| < \delta$ then $|f(z) - f(z^{'})| < \epsilon$. Now, suppose α and β on the line segment $[c, a]$ and $[b, c]$ respectively, so that $|\alpha - a| < \delta$ and $|\beta - b| < \delta$. Let T_1=$[\alpha, \beta, c, \alpha]$ and Q= $[a, b, \beta, \alpha, a]$. Then $\int_T f = \int_{T_1} f + \int_Q f$, but T and its inside are contained in G_+ and f is analytic there; hence

$$\int_T f = \int_Q f \tag{4.1}$$

since $\int_{T_1} f = 0$ by C.I.T, But, if $0 \leq t \leq 1$, then

$$|[t\beta + (1-t)\alpha] - [tb + (1-t)a]| < \delta$$

so that,

$$|f(t\beta + (1-t)\alpha) - f(tb + (1-t)a)| < \epsilon$$

If $M = max\{|f(z)| : z \in \triangle\}$ and l = the perimeter of T, then

$$\begin{aligned}
|\int_{[a,b]} f + \int_{[\beta,\alpha]} f| &= |(b-a)\int_0^1 f(tb + (1-t)a)dt \\
& \quad -(\beta - \alpha)\int_0^1 f(t\beta + (1-t)\alpha)dt| \\
&\leq |b-a||\int_0^1 [f(tb + (1-t)a) \\
&- \quad f(t\beta + (1-t)\alpha)]dt| + \\
& \quad |(b-a) - (\beta - \alpha)| \\
& \quad |\int_0^1 f(t\beta + (1-t)\alpha)dt| \\
&\leq \epsilon|b-a| + M|(b-\beta) + (\alpha - a)| \\
&\leq \epsilon l + 2M\delta.
\end{aligned}$$

Also,

$$(4.2) \qquad |\int_{[\alpha,a]} f| \leq M|a-\alpha| \leq M\delta.$$

and

$$(4.3) \qquad |\int_{[b,\beta]} f| \leq M\delta.$$

Combining these last two inequalities with (4.1) and (4.2) gives that

$$|\int_T f| \leq \epsilon l + 4M\delta.$$

Since it is possible to choose$\delta < \epsilon$ and since ϵ is arbitrary, it is follows that

$$|\int_T f| = 0$$

Thus f must be analytic. This completes the proof. □

3. Analytic Continuation Along a Path

Let us begin this section by recalling the definition of a function. We use the somewhat imprecise statement that a function is a triple (f, G, Ω), where G and Ω are sets and f is a rule which assigns to each element of G a unique element of Ω. If we enlarge the range Ω to a set Ω_1 then (f, G, Ω_1) is a different function. However this point should not be emphasized here; we do emphasize that a change in a domain results in a new function. Indeed, the purpose of analytic continuation is to enlarge the domain.

Definition 4.2. Germ: A function element is a pair (f, G) where G is a region and f is analytic function on G. For a given function elements (f, G) define the germ of f at a to be the collection of all function elements (g, D) such that $a \in \mathbb{D}$ and $f(z) = g(z)$ for all z in a neighborhood of a. Denote the germ by $[f]_a$. Notice that $[f]_a$ is a collection of all function elements and it is not a function element itself.

DEFINITION 4.3. Let $\gamma[[0,1] \longrightarrow \mathbb{C}$ be a path and suppose that for each t in [0,1] there is a function element $(f_t, \mathbb{D}_t)$ such that :

(a) $\gamma(t) \in \mathbb{D}_t$;

(b) for each t in [0, 1] there is a $\delta > 0$ such that $|s-t| < \delta$ implies $\gamma(s) \in \mathbb{D}_t$ and $[f_s]_{\gamma(s)}$=$[f_t]_{\gamma(s)}$. Then $(f_t, \mathbb{D}_t)$ is the analytic continuation of $(f_0, \mathbb{D}_0)$ along the path γ; or $(f_1, \mathbb{D}_1)$ is obtained from $(f_0, \mathbb{D}_0)$ by analytic continuation along γ.

PROPOSITION 4.4. *Let $\gamma : [0,1] \to \mathbb{C}$ be a path from a to b and let $\{(f_t, D_t) : 0 \leq t \leq 1\}$ and $\{(g_t, B_t) : 0 \leq t \leq 1\}$ be analytic continuations along γ such that $[f_0]_a = [g_0]_a$.Then $[f_1]_b = [g_1]_b$.*

PROOF. This proposition will be proved by showing that the set $T = \{t \in [0,1] : [f_t]_{\gamma(t)} = [g_t]_{\gamma(t)}\}$ is both open and closed in [0,1] so that, in particular, $1 \in T$. The easiest part of the proof is to show that T is open. So fix t in T and assume $t \neq 0$ or 1. (If $t = 1$ the proof is complete; if $t = 0$ then the argument about to be given will also show that $[a, a+\delta) \subset T$ for some $\delta > 0$.) By the definition of analytic continuation there is a $\delta > 0$ such that for all $|s-t| < \delta$, $\gamma(s) \in D_t \cap B_t$ and

$$[f_s]_{\gamma(s)} = [f_t]_{\gamma(s)}$$
$$[g_s]_{\gamma(s)} = [g_t]_{\gamma(s)}.$$

But since $t \in T$, $f_t(z) = g_t(z)$ for all z in $D_t \cap B_t$. So it follows from the above equation that $[f_s]_{\gamma(s)} = [g_s]_{\gamma(s)}$ whenever $|s-t| < \delta$. That is, $(t-\delta, t+\delta) \subset T$ and so T is open. To show that T is closed let t be a limit point of T, and again choose $\delta > 0$ so that $\gamma(s) \in D_t \cap B_t$ and the above equation is satisfied when ever $|s-t| < \delta$. Since t is a limit point of T there is a point s in T with $|s-t| < \delta$; so $G = D_t \cap B_t \cap D_s \cap B_s$ contains $\gamma(s)$ and therefore, is a non-empty open set. Thus, $f_s(z) = g_s(z)$ for all $z \in G$ by definition of T. But, according to the above equation $f_s(z) = f_t(z)$ and $g_s(z) = g_t(z)$ for all z in G. So $f_t(z) = g_t(z)$ for all z in G and, because G has a limit point in $D_t \cap B_t$, this gives that $[f_t]_{\gamma(t)} = [g_t]_{\gamma(t)}$. That is, $t \in T$ and so T is closed. $\square$

Definition 4.5. If $\gamma : [0,1] \to \mathbb{C}$ is a path from a to b and $\{(f_t, D_t) : 0 \leq t \leq 1\}$ is an analytic continuation along γ then the germ $[f_1]_b$ is the analytic continuation of $[f_0]_a$ along γ.

Definition 4.6. If (f, G) is a function element then the complete analytic function obtained from (f, G) is the collection $\mathbb{F}$ of all germs $[g]_b$ for which there is a point a in G and a path γ from a to b such that $[g]_b$ is the analytic continuation of $[f]_a$ along γ.

Definition 4.7. Unrestricted Analytic Continuation Let (f, D) be a function element and let G be a region which contains D; then (f, D) admits unrestricted analytic continuation in G if for any path γ in G with initial point in D there is an analytic continuation of (f, D) along γ.

Theorem 4.8 (Monodromy Theorem). *Let (f, D) be a function element and let G be a region containing D such that (f, D) admits unrestricted continuation in G.*
Let $a \in D$, $b \in G$ and let γ_0 and γ_1 be paths in G from a to b; let $\{(f_t, D_t) : 0 \leq t \leq 1\}$ and$\{(g_t, D_t) : 0 \leq t \leq 1\}$ be analytic continuations of (f, D) along γ_0 and γ_1 respectively.
If γ_0 and γ_1 are FEP(fixed-end-point) homotopic in G then $[f_1]_b = [g_1]_b$.

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